1. Introduction

In this project, we design a Quadrature Mirror Filter (QMF) Bank with application to sub-band image coding. Then we extend our result to four-band filter bank.

In section 2 we design a QMF filter bank with the choice of analysis filters \( H_1(z) = H_0(-z) \). We then find \( H_0(z), H_1(z), F_0(z), F_1(z) \) so that our system satisfies perfect reconstruction constraint. We will see simulation result and advantages and disadvantages of such a system through simulation.

In section 3 we want \( H_0(z) \) to be a linear-phase FIR filter. We design our filters so we again meet perfect reconstruction constraint. We will see properties of such a filter.

In section 4 we design a perfect reconstruction QMF bank with IIR analysis and synthesis filters. We simulate the system and see pros and cons of the design.

Now we test the results of previous sections and their applications on image sub-band coding. In section 5 we extend the concept of 1-D filter banks to 2-D. We apply our filters to some sample images and try to get as higher compression rate as possible by assigning less number of bits to bands with less energy level. We develop a performance evaluation measure so that we can compare different filter banks.

Section 6 extends previous experiment to four-band filter bank. We develop our 2-D filter banks so we can get better performance in sense of less number of bits per pixel along with preserving image quality to a reasonable extent.

2. Two Channel Perfect Reconstruction Filter bank

A diagram for the analysis/synthesis system used for sub-band coding with two channels is shown in Figure 2.1
The relation between input $x[n]$ and output $\hat{x}[n]$ in $z$ domain is defined as

$$\hat{X}(z) = \frac{1}{2} [H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) + \frac{1}{2} [H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z)$$

(2.1)

This can be written as

$$\hat{X}(z) = T(z)X(z) + A(z)X(-z)$$

(2.2)

which $T(z)$ is the distortion transfer function and $A(z)$ is the aliasing term.

To achieve perfect reconstruction, $T(z)$ must be pure delay and $A(z)$ must be canceled completely. That is the filters must satisfy the following conditions:

$$\frac{1}{2} [H_0(z)F_0(z) + H_1(z)F_1(z)] = k z^{-n_0}$$

(2.3)

$$\frac{1}{2} [H_0(-z)F_0(z) + H_1(-z)F_1(z)] = 0$$

(2.4)

Solving the above equations for $F_0(z)$ and $F_1(z)$ we obtain

$$F_0(z) = \frac{2k z^{-n_0} H_1(-z)}{H_0(z)H_1(-z) - H_0(-z)H_1(z)}$$

(2.5)

$$F_1(z) = \frac{-2k z^{-n_0} H_0(-z)}{H_0(z)H_1(-z) - H_0(-z)H_1(z)}$$

(2.6)

This is the most general constraint that we have on our synthesis filters so that we can achieve perfect reconstruction. This will be used in later sections.

**2.1 Quadrature Mirror Filter Bank**
If we constrain $H_0(z)$ and $H_1(z)$ to be mirror image of each other,

i.e. $H_1(z) = H_0(-z)$

then we have

$$F_0(z) = \frac{2kz^{-\beta_0}H_0(z)}{H_0^2(z) - H_0^2(-z)}$$  \hspace{1cm} (2.7)

$$F_1(z) = \frac{-2kz^{-\beta_0}H_0(-z)}{H_0^2(z) - H_0^2(-z)}$$  \hspace{1cm} (2.8)

So, if we design filter $H_0(z)$ we are able to design other filters according to equations (2.7), (2.8), and $H_1(z) = H_0(-z)$. These filters can be FIR or IIR.

Also, we can put another constrain and consider only designing with FIR filters. Consequently the two following conditions must be satisfied by $H_0(z)$

$$H_0(z) + H_0(-z) = \alpha_0 z^{-\beta_0}$$ \hspace{1cm} (2.9)

$$H_0(z) - H_0(-z) = \alpha_i z^{-\beta_i}$$ \hspace{1cm} (2.10)

where $\alpha_0$ and $\alpha_i$ are real constants and $\beta_0$ and $\beta_i$ are integers. Solving the equations we result

$$H_0(z) = \frac{1}{2} \alpha_0 z^{-\beta_0} + \frac{1}{2} \alpha_i z^{-\beta_i}$$ \hspace{1cm} (2.11)

$$H_0(-z) = \frac{1}{2} \alpha_0 z^{\beta_0} - \frac{1}{2} \alpha_i z^{\beta_i}$$ \hspace{1cm} (2.12)

Having the above equations satisfied, we need that $\beta_0$ be even and $\beta_i$ be odd integers so we define them as

$$\beta_0 = 2n_0$$

$$\beta_i = 2n_i + 1$$

Also we define

$$c_0 = \frac{1}{2} \alpha_0$$

$$c_i = \frac{1}{2} \alpha_i$$

Thus the general form for quadratic mirror two channel FIR filters will be
\[
\begin{align*}
H_0(z) &= c_0 z^{-2n_0} + c_1 z^{-2n_1+1} \\
H_1(z) &= c_0 z^{-2n_0} - c_1 z^{-2n_1+1}
\end{align*}
\]  

(2.13) 

(2.14)

Clearly the filters are from order one so they cannot be good approximations for ideal low-pass and high pass filters. The frequency response of the filters are shown in Figure 2.2 for \( c_0 = c_1 = \frac{1}{2} \) and \( n_0 = n_1 = 0 \).

![Figure 2.2](image)

**Figure 2.2** – Frequency response of \( H_0(z) \) and \( H_1(z) \) defined in (2.13) and (2.14) for \( c_0 = c_1 = \frac{1}{2} \) and \( n_0 = n_1 = 0 \)

### 2.2. Simulation

We can test perfect reconstruction property of the system through simulation.
A random input applied to the system of figure 2.1 with filters $H_0(z)$ and $H_1(z)$ is shown in figure 2.2. The simulation result can be seen in figure 2.2. As depicted in figure, the two plots of $x(n)$ and $\hat{x}(n)$ are identical (except one step delay) which shows the system is perfect reconstruction.

2.3. Discussion

Making assumption of $H_1(z) = H_0(-z)$ in filter design has advantages and disadvantages. As we can see in 2.7 and 2.8, synthesis filters $F_0(z)$ and $F_1(z)$ and analysis filter $H_1(z)$ all can be designed easily when $H_0(z)$ is designed. So this makes designing process fast and easy. Meanwhile we face some quality degradation using such a system.

If we use FIR filters, having system perfect reconstruction makes degree of freedom in designing the filters low (2.13 and 2.14) and such a system does not have good quality and we cannot achieve sharp cut-off. Thus frequency bands cannot be separated very well.

If we use IIR filters, we may face stability problems on $F_0(z)$ and $F_1(z)$. Hence, finding stable filters is so hard and such design is not trivial. We will see more on section 4.
3. Linear-phase FIR filter Design

In this section we are about to design \( H_0(z) \) and \( H_1(z) \) as a linear-phase FIR filter. As shown in previous section, having the assumption \( H_1(z) = H_0(-z) \) we can design a perfect reconstruction system, but such a system does not have acceptable response in stop and pass bands, and as a result we do not get good quality and performance. So, to achieve better performance we need to relax the condition \( H_1(z) = H_0(-z) \) so we can design \( H_0(z) \) and \( H_1(z) \) separately yet preserving perfect reconstruction condition. As you will see in the following subsection, designing such a filter is not easy but will give us a really better performance than the order one filter we came to in previous section.

3.1. Designing a High Quality FIR Perfect Reconstruction Filter Bank

If we relax the condition \( H_1(z) = H_0(-z) \), it is possible to design \( H_0(z) \) and \( H_1(z) \) with sharp cut-off band. The cost we must pay for such filters is the complexity in design and implementation of high order filters. Here we will introduce a two-step procedure to design a perfect reconstruction two-channel filter bank. In the first step we obtain the impulse response of \( H_1(z) \) to satisfy the requirements of cut-off frequency and transition band. In the second step we will find \( H_0(z) \) both have acceptable frequency response and satisfy the condition of perfect reconstruction.

3.1.1. Designing \( H_1(z) \)

We assume that \( H_1(z) \) is a linear FIR filter from an even order i.e.

\[
h[n] = h[N_1 - n] \quad 0 \leq n \leq N_1
\]  

where \( N_1 \) is any even integer. The Z-transform of \( h[n] \) will be

\[
H_1(z) = \sum_{n=0}^{N_1} h[n]z^{-n}
\]  

We can show that the Fourier transform of \( h[n] \) can be written as

\[
H_1(e^{j\omega}) = e^{-jN_1\omega/2} \left[ h[N_1/2] + 2 \sum_{n=0}^{N_1/2-1} h[n] \cos(N_1/2 - 1)\omega \right]
\]  

Let us define \( g_k \) as
\[ g_k = \begin{cases} h_i[N_i/2] & k = 0 \\ 2h_i[N_i/2-k] & k = 1, 2, \ldots, N_i/2 \end{cases} \] (3.4)

then we have

\[ H_i(e^{j\omega}) = e^{-jN_i\omega/2} \sum_{k=0}^{N_i/2} g_k \cos(k\omega) \] (3.5)

Also we define

\[ \mathbf{g} = \begin{bmatrix} g_0 & g_1 & \ldots & g_{M_i} \end{bmatrix}^T \] (3.6)

\[ \mathbf{c}_i(\omega) = \begin{bmatrix} 1 & \cos\omega & \ldots & \cos(M_i\omega) \end{bmatrix}^T \] (3.7)

where \( M_i = N_i/2 \). Based on the above definitions we can write

\[ |H_i(e^{j\omega})| = \mathbf{c}_i(\omega)^T \mathbf{g} \] (3.8)

According to figure 3.1 we denote by \( \omega_s \) and \( \omega_p \), the stop and pass bands of \( H_i(z) \).

\[ \text{Figure 3.1 – Stop and pass bands in a high pass filter} \]

The total power of error (deviation of \( |H_i(e^{j\omega})| \) from ideal filter) in stop and pass bands are

\[ E_s = \int_{0}^{\omega_s} |H_i(e^{j\omega})|^2 \, d\omega \] (3.9)

\[ E_p = \int_{\omega_p}^{\pi} \left( |H_i(e^{j\omega})| - |H_i(e^{j\pi})| \right)^2 \, d\omega \] (3.10)

Considering (3.8), one can show that \( E_s \) and \( E_p \) can be written in the following form
\[ E_{n} = g^\text{T} P_{1} g \]  
\[ E_{n} = g^\text{T} Q_{1} g \]  

where \( P_{1} \) and \( Q_{1} \) are defined as

\[ P_{1} = \int_{\omega_{s}}^{\omega_{l}} c_{1}(\omega)c_{1}^\text{T}(\omega) d\omega \]  
\[ Q_{1} = \int_{\omega_{s}}^{\omega_{l}} (c_{1}(\omega) - c_{1}(\pi))(c_{1}(\omega) - c_{1}(\pi))^\text{T} d\omega \]

Now we define the cost function of our optimization problem as a linear combination of \( E_{n} \) and \( E_{n} \)

\[ J = \alpha E_{n} + (1 - \alpha) E_{n} \]  

Defining \( R_{1} \) as

\[ R_{1} = \alpha P_{1} + (1 - \alpha) Q_{1} \]  

We can write the cost function as

\[ J = g^\text{T} R_{1} g \]  

Our goal in this problem is to obtain \( g \) such that \( J \) be minimized and also the frequency response of the filter be equal to 1 at \( \omega = \pi \). The solution to this problem is

\[ g = \frac{v_{m}}{c_{1}^\text{T}(\pi)v_{m}} \]  

where \( v_{m} \) is the eigenvector of \( R_{1} \) corresponding to smallest eigen-value.

Note that \( \alpha \) is a factor, which specify the relative importance of the error in stop and pass bands.

By proper choice of the parameters \( \alpha \), \( \omega_{s} \), and \( \omega_{l} \) we can obtain the best performance of the filter \( H_{0}(z) \).

3.1.2. Designing \( H_{0}(z) \)
The design of $H_0(z)$ is more difficult than $H_1(z)$. In this case not only we must design $H_0(z)$ for acceptable frequency response, but we must meet the perfect reconstruction requirements. Again we assume that $H_0(z)$ is linear phase with even order $N_0$. Let

\[ h_0[n] = h_0[N_0 - n] \quad 0 \leq n \leq N_0 \quad (3.18) \]

and assume that

\[ N_0 = N_1 + 4L - 2 \quad (3.19) \]

We will see that $L$ is the number of free parameters we can use for shaping the frequency response of $H_0(z)$. The remaining degrees of freedom ($N_1/2 + L$) will be used to satisfy the perfect reconstruction requirements. The poly-phase components of $H_0(z)$ and $H_1(z)$ can be written as

\begin{align*}
E_{00}(z) &= \sum_{k=0}^{N_1/2+2L-4} h_0[2k]z^{-k} \\
E_{01}(z) &= \sum_{k=0}^{N_1/2+2L-2} h_0[2k+1]z^{-k} \\
E_{10}(z) &= \sum_{k=0}^{N_1/2} h_1[2k]z^{-k} \\
E_{11}(z) &= \sum_{k=0}^{N_1/2-1} h_1[2k+1]z^{-k}
\end{align*}

And the poly-phase matrix will be

\[ E(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix} \quad (3.24) \]

We define $F(z)$ as the determinant of $E(z)$ and we know that $F(z)$ is from order $N_1 + 2L - 2$. Let us define vectors $f$ and $h_0$ as

\begin{align*}
f &= [f[0] \quad f[1] \quad \ldots \quad f[N_1 + 2L - 2]] \\
h_0 &= [h_0[0] \quad h_0[1] \quad \ldots \quad h_0[N_1 + 4L - 2]]
\end{align*}

Also we define the $(N_1 + 2L - 1) \times (N_1 + 4L - 1)$ matrix $H_1$ as follows
\[
\begin{bmatrix}
    h[1] & -h[0] & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    h[N_i - 1] & -h[N_i - 2] & \ldots & h[1] & -h[0] & \ldots & 0 & 0 & 0 & \ldots & 0 \\
    0 & -h[N_i] & h[N_i - 1] & -h[N_i - 2] & \ldots & h[1] & -h[0] & 0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
    0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -h[N_i] & h[N_i - 1]
\end{bmatrix}
\]

(3.28)

Note that the vector \( \mathbf{h} \) was already computed so the matrix \( \mathbf{H}_1 \) is completely known. Based on definition of \( \mathbf{f} \), \( \mathbf{h}_0 \), and \( \mathbf{H}_1 \) we know

\[
\mathbf{f} = \mathbf{H}_1 \mathbf{h}_0
\]

(3.29)

Because \( h[0] \) and \( h[1] \) are both linear phase it is easy to show \( f[n] \) is also linear phase i.e.

\[
f[n] = f[N_i + 2L - 2 - n] \quad 0 \leq n \leq N_i + 2L - 2
\]

(3.30)

Thus the knowledge of \( \mathbf{f} \) is equivalent to knowledge of

\[
\tilde{\mathbf{f}} = \begin{bmatrix} f[0] & f[1] & \ldots & f[N_i/2 + L - 1] \end{bmatrix}
\]

(3.31)

Defining the \((N_i/2 + L) \times (N_i + 2L - 1)\) matrix \( \tilde{\mathbf{F}} \) as

\[
\tilde{\mathbf{F}} = \begin{bmatrix} \mathbf{I}_{N_i/2 + L} & \mathbf{0}_{(N_i/2 + L) \times (N_i/2 + L - 1)} \end{bmatrix}
\]

(3.32)

we see

\[
\tilde{\mathbf{f}} = \tilde{\mathbf{F}} \mathbf{H}_1 \mathbf{h}_0
\]

(3.33)

Note that the components of \( \mathbf{f} \) are symmetric with respect to \( f[N_i/2 + L - 1] \), also for perfect reconstruction just one component of \( \mathbf{f} \) is nonzero. Thus the only way that the perfect reconstruction can hold is that

\[
f[n] = \begin{cases} 
    0 & n \neq N_i/2 + L - 1 \\
    d & n = N_i/2 + L - 1
\end{cases}
\]

(3.34)

where \( d \) is an arbitrary constant. The above condition is equivalent to

\[
\tilde{\mathbf{f}} = de
\]

(3.35)
where $e$ is a $(N_0/2 + L) \times 1$ vector defined as

$$e = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}^T$$

(3.36)

Thus the perfect reconstruction condition can be cast in the form

$$(\tilde{F}H)h_0 = de$$

(3.37)

Based on (3.33) and (3.35) we define the vector $r$ as

$$r = \begin{bmatrix} r_0 & r_1 & \ldots & r_{N_0/2} \end{bmatrix}$$

(3.38)

where

$$r_k = \begin{cases} h_0[N_0/2] & k = 0 \\ 2h_0[N_0/2 - k] & k = 1, 2, \ldots, N_0/2 \end{cases}$$

(3.39)

We also define the $(N_0 + 1) \times (N_0/2 + 1)$ matrix $S$ as

$$S = \begin{bmatrix} 0 & 0 & 0 & \ldots & 1/2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1/2 & \ldots & 0 \\ 0 & 1/2 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1/2 & 0 & \ldots & 0 \\ 0 & 0 & 1/2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

(3.40)

Clearly

$$h_0 = Sr$$

(3.41)

Plugging the above equation in 3.37 we can write the perfect reconstruction condition in the following way

$$\tilde{F}H, Sr = de$$

(3.42)

Defining $\tilde{H}$ as

$$\tilde{H} = \tilde{F}H, S$$

(3.43)
we can write (3.42) as

\[ H_1 r = d e \]  

(3.44)

Now we focus on using the remaining degrees of freedom for shaping the frequency response of 
\( H_0(z) \) as a low-pass filter. By \( c_0(\omega) \) defined as

\[
c_0(\omega) = \begin{bmatrix}
1 & \cos \omega & \cdots & \cos \left( \frac{N_0}{2} \omega \right)
\end{bmatrix}^T
\]

(3.45)

the frequency response of \( H_0(z) \) is written as

\[
|H_0(e^{j\omega})| = c_0^T(\omega)r
\]

(3.46)

Consider the frequency response of a low pass filter

![Figure 3.2 – Stop and pass bands for a low pass filter](image)

Based on \( \omega_s \) and \( \omega_p \) shown in this figure, we define the power of error is pass band an stop band as

\[
E_{s_0} = \int_{\omega_s}^{\omega_0} |H_0(e^{j\omega})|^2 d\omega
\]

(3.47)

\[
E_{p_0} = \int_{0}^{\omega_p} \left( |H_0(e^{j\omega})| - |H_0(e^{j0})| \right)^2 d\omega
\]

(3.48)

Like the last section we can see

\[
E_{s_0} = r^T P_s r
\]

(3.49)

\[
E_{p_0} = r^T Q_p r
\]

(3.50)
where

\[
P_0 = \int_{\omega_o}^{\omega_0} c_0(\omega) e_0^T(\omega) d\omega
\]

(3.51)

\[
Q_0 = \int_{0}^{\omega_0} (c_0(\omega) - c_0(0))(e_0(\omega) - e_0(0))^T d\omega
\]

(3.52)

We define the matrix \( R_0 \) as a linear combination of \( P_0 \) and \( Q_0 \) that is

\[
R_0 = \beta P_0 + (1 - \beta) Q_0
\]

(3.53)

We recall that \( \beta \) is a parameter to trade-off between the error of pass band and the stop band.

The cost function we will use in this case is

\[
J = r^T R_0 r
\]

(3.54)

and our goal is to minimize this cost function subject to the constrain (3.44) that is

\[
\min_{r} \quad r^T R_0 r
\]

s.t. \( H_r r = d e \)

(3.55)

Using the method of Lagrange multipliers the solution of this optimization problem will be

\[
r = d R_0^T \tilde{H}_r^T (\tilde{H}_r R_0^T \tilde{H}_r^T)^{-1} e
\]

(3.56)

Also we determine \( d \) such that

\[
\left| H_0(e^{j0}) \right| = 1
\]

(3.57)

which results

\[
d = \frac{1}{c_0(0) R_0 \tilde{H}_r^T (\tilde{H}_r R_0 \tilde{H}_r^T)^{-1} e}
\]

(3.58)

Finally we obtain \( h_0 \) using (3.41)

\[
h_0 = \frac{S R_0 \tilde{H}_r^T (\tilde{H}_r R_0 \tilde{H}_r^T)^{-1} e}{c_0(0) R_0 \tilde{H}_r^T (\tilde{H}_r R_0 \tilde{H}_r^T)^{-1} e}
\]

(3.59)

Having \( H_0(z) \) and \( H_1(z) \) determined, we obtain \( F_0(z) \) and \( F_1(z) \) as follows
\[ R(z) = z^{-m_0} E^{-1}(z) = \frac{z^{-m_0}}{d} z^{-(N_1/2 + L - 1)} \left[ \begin{array}{cc} E_{11}(z) & -E_{01}(z) \\ -E_{10}(z) & E_{00}(z) \end{array} \right] \]  

(3.60)

where \( m_0 \) is any positive integer. With \( m_0 = N_1/2 + L - 1 \) we obtain

\[ F_0(z) = \frac{1}{d} \left[ -E_{10}(z^2) + z^{-1}E_{11}(z^2) \right] \]  

(3.61)

\[ F_1(z) = \frac{1}{d} \left[ E_{00}(z^2) - z^{-1}E_{01}(z^2) \right] \]  

(3.62)

and the distortion function will be

\[ T(z) = z^{-(N_1/2 + L - 1)} \]  

(3.63)

Based on MATLAB a m-file is generated to perform all the computations. The source of this file is included at the end of the report. Using this computer program for the following values we obtained \( H_0(z) \), \( H_1(z) \), \( F_0(z) \), and \( F_1(z) \).

\[ N_1 = 30 \]

\[ L = 6 \]

\[ \alpha = 0.95 \]

\[ \beta = 0.9 \]

\[ \omega_{n_1} = 9\pi/20 \]

\[ \omega_{n_1} = 11\pi/20 \]

\[ \omega_{n_1} = 3\pi/5 \]

\[ \omega_{n_1} = 2\pi/5 \]

The resultant impulse responses are given at the end of the report. Also the frequency response of the filters are illustrated in figure 3.3.
3.2. Discussion

In the last subsection we presented a procedure to design perfect reconstruction two-channel filter bank. This method can always be used for designing the filter banks. Also as it can be seen in figure 3.3 the frequency response of filters approximate the ideal case well, the frequency response in pass band is almost flat and in stop band the filters have more than 40 dB attenuation. The disadvantage of this method is that their order is relatively high that is $N_i = 30$ and $N_s = 52$.

4. IIR filter Design

In this section we discuss the possible application of IIR filters in filter banks, assuming that the condition $H_i(z) = H_0(-z)$ holds. The main difficulty of this approach is that although stable $H_0(z)$ and $H_i(z)$ can be easily designed, it is possible that $F_0(z)$ and $F_i(z)$ obtained by (2.7) and (2.8) be unstable. One method to overcome the difficulty is to design $H_0(z)$ containing one or more parameters and then try to find the parameter(s) such that $F_0(z)$ and $F_i(z)$ be stable. In the remaining of this section we will show the method for a second order IIR $H_0(z)$. 
One method for designing a digital filter is to first design an analog counterpart, then using bilinear transform to obtain the discrete version of that analog filter. The bilinear transform defined by

\[ s = \alpha \frac{1 - z^{-1}}{1 + z^{-1}} \]  

(4.1)

has the advantage that it maps the inside of unit circle to the left half of s-plan. This property guarantees that under the bilinear transformation, a stable analog filter results a stable digital filter.

We use the Butterworth analog filters as the bases of our design. The transfer function for a second order Butterworth filter is

\[ H_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \]  

(4.2)

Applying the bilinear transform to the above analog filter results

\[ H_b(z) = \frac{1 + 2z^{-1} + z^{-2}}{(\alpha^2 + \sqrt{2}\alpha + 1) + 2(1 - \alpha^2)z^{-1} + (\alpha^2 - \sqrt{2}\alpha + 1)z^{-2}} \]  

(4.3)

Let us define

\[ A(z) = 1 + 2z^{-1} + z^{-2} \]  

(4.4)

\[ B(z) = (\alpha^2 + \sqrt{2}\alpha + 1) + 2(1 - \alpha^2)z^{-1} + (\alpha^2 - \sqrt{2}\alpha + 1)z^{-2} \]  

(4.5)

We obtain from (2.7) and (2.8)

\[ F_0(z) = \frac{-2kz^{-m_0}A(z)B(z)B^2(-z)}{A^2(z)B^2(-z) - A^2(-z)B^2(z)} \]  

(4.6)

\[ F_1(z) = \frac{-2kz^{-m_0}A(-z)B(-z)B^2(z)}{A^2(z)B^2(-z) - A^2(-z)B^2(z)} \]  

(4.7)

The stability of filters depends on the location of roots of

\[ D(z) = A^2(z)B^2(-z) - A^2(-z)B^2(z) \]  

(4.8)

At this point we try to find a proper value for \( \alpha \) such that all roots of \( D(z) \) be located inside the unit circle. Using a computer program we compute the roots of \( D(z) \) for different values of \( \alpha \). The resultant root locus is plotted in figure 4.1 Note that the pole at \( z^{-1} = 0 \) cannot be counted as an unstable pole, because by proper choice of \( m_0 \) we can cancel it.
Figure 4.1 – Root locus of \( D(z) \)

Clearly for some values of \( \alpha \ (\alpha < 1) \), all roots of \( D(z) \) are inside the unit circle and filters are stable. On the other hand for \( \alpha > 1 \) the filters have to unstable poles.

For \( \alpha = 0.9 \), the transfer functions of \( H_0(z) \) and \( H_1(z) \) are

\[
H_0(z) = \frac{1 + 2z^{-1} + z^{-2}}{3.08 + 0.38z^{-1} + 0.54} \tag{4.9}
\]

\[
H_1(z) = \frac{1 - 2z^{-1} + z^{-2}}{3.08 - 0.38z^{-1} + 0.54} \tag{4.10}
\]

The frequency response of \( H_0(z) \) and \( H_1(z) \) for \( \alpha = 0.9 \) is illustrated in figure 4.2. Also to make sure that the whole system is perfect reconstruction, the distortion function \( T(z) \) is shown in figure 4.3.
Figure 4.2 – Frequency response of IIR Filters

Figure 4.3 - Distortion function for IIR filter bank
4.1. Discussion

In general with equal degrees, IIR filters can better approximate an ideal filter than FIR filters. But as it is illustrated by figure 4.2, the frequency response of $H_0(z)$ and $H_1(z)$ are not completely satisfactory. The reason is the low degree of them. Based of the method we discussed here, it is not easy to design a high order IIR filter. In other word the method used here cannot be easily extended to higher order filters. Again we recall that the main difficulty of using IIR filters in perfect reconstruction filter banks is the stability of synthesis filters.

5. Two dimensional QMF and image coding

In this section we apply the filters designed in previous sections to a practical experiment. We are to perform image coding by taking advantage of sub-band coding so that we can compress images as much as possible. The idea is that usually most of the energy of images is located in lower band frequencies. So if we separate the frequency bands, we are able to designate less number of bits to higher frequency portion of the image. So, totally we receive less number of bits per pixel.

For applying designed 1-D filters to 2-D case, we need an extension. As depicted in figure 5.1, we can divide frequency bands in the two dimensions and perform coding of each band separately (figure 5.1). We apply 1-D filters once on rows and next on columns of image.

![Figure 5.1 – Sub-band Coding of a 2-D Image](image)

In the following three subsections, we apply three filter banks designed in sections 2, 3 and 4 and see their performances. These filters will be extended to 4-band filter banks in section 6. All simulations were done in Matlab.

5.1. FIR filter with $H_1(z) = H_0(-z)$

The filter designed in section 2 was applied to two sample benchmarks (lenna and baboon). Since system is perfect reconstruction, the output of the system is absolutely identical to the original file. (figures 5.1 and 5.3)

The main characteristic of image files is that most of the energy is centralized in lower frequency band. This can be seen from figures 5.2 and 5.4 where different frequency elements of the signal were separated. Also, the energy level of each band is reported in Table 5.1. These all
show that we can easily reduce the number of designated bits to higher bands without losing considerable energy. In fact these degradations cannot be detected easily by human’s eye. The interesting issue is that this degradation of coding is negligible as compared to quantization error, which is unavoidable.

Figure 5.1 – Left: Original Image (lenna.tiff) scale by 40% , Right: Output of filter designed in section 2

Figure 5.2 – Sub-bands of Image shown in figure 5.1 using filter banks designed in section 2 (all but first one are gained by 10 to make them more visible)
Figure 5.3 – Left: Original Image (baboon.tiff) scale by 40%, Right: Output of filter designed in section 2

Figure 5.4 – Sub-bands of Image shown in figure 5.3 using filter banks designed in section 2 (all but first one are gained by 10 to make them more visible)
<table>
<thead>
<tr>
<th></th>
<th>Lenna</th>
<th>Baboon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>99.5109%</td>
<td>98.0257%</td>
</tr>
<tr>
<td></td>
<td>0.2941%</td>
<td>0.5109%</td>
</tr>
<tr>
<td></td>
<td>0.1454%</td>
<td>1.1626%</td>
</tr>
<tr>
<td></td>
<td>0.0496%</td>
<td>0.3008%</td>
</tr>
</tbody>
</table>

Table 5.1 – Energy percentage of different sub-bands divided by filter designed in section 2

5.2. FIR filter with no constraint

In next experiment the filter designed in section three was applied to same images and similar results were received. It shows that since in these images, most of the energy (more than 98%) is in first quarter of frequency spectrum, even if a very simple FIR filter is applied, we can achieve very good performance and high compression result. (figures 5.5 to 5.8 and Table 5.2)

Figure 5.5 - Left: Original Image (lenna.tiff) scale by 40% , Right: Output of FIR filter designed in section 3
Figure 5.6 – Sub-bands of Image shown in figure 5.5 using FIR filter banks designed in section 3 (all but first one are gained by 10 to make them more visible)

Figure 5.7 – Left: Original Image (baboon.tiff) scale by 40% , Right: Output of FIR filter designed in section 3
Figure 5.8 – Sub-bands of Image shown in figure 5.7 using FIR filter banks designed in section 3 (all but first one are gained by 10 to make them more visible)

<table>
<thead>
<tr>
<th></th>
<th>Lenna</th>
<th>Baboon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy percentage</td>
<td>99.4105 %</td>
<td>98.0630 %</td>
</tr>
<tr>
<td>0.0991 %</td>
<td>0.0460 %</td>
<td>1.1435 %</td>
</tr>
<tr>
<td>0.1967 %</td>
<td>0.4085 %</td>
<td>0.2725 %</td>
</tr>
</tbody>
</table>

Table 5.2 – Energy percentage of different sub-bands divided by filter designed in section 3

In figure 5.9 the relation between energy of error signal and number of bits per pixel was shown. As it can be seen, by using less than 1 bit per pixel we can almost get complete performance and error level is almost negligible. Output images stored using only 1.5 bits per pixel can be seen in figure 5.10 which is almost same as original images. Similar results can be achieved from IIR filter which we preferred not to repeat.
6. The 4-Channel Filter Bank

Based on the design of a 2-channel perfect reconstruction filter bank, we can design a 4-channel bank with perfect reconstruction property. Here is the main idea behind this design. Assume that we have applied the input signal to a 2-channel bank. Now we apply the output of each channel to the input of another bank with exactly the same design. At this stage we have four outputs with rate of $1/4$. Let us denote by $\overline{H}_k(z) \quad k = 0, 1, 2, 3$ the transfer function of the new analysis bank then
\[
\begin{align*}
\mathcal{H}_0(z) &= H_0(z)H_0(z^2) \\
\mathcal{H}_1(z) &= H_0(z)H_1(z^2) \\
\mathcal{H}_2(z) &= H_1(z)H_0(z^2) \\
\mathcal{H}_3(z) &= H_1(z)H_1(z^2) \\
\end{align*}
\] 

(6.1)

We can use the same method to determine the transfer functions of synthesis bank

\[
\begin{align*}
\mathcal{F}_0(z) &= F_0(z)F_0(z^2) \\
\mathcal{F}_1(z) &= F_0(z)F_1(z^2) \\
\mathcal{F}_2(z) &= F_1(z)F_0(z^2) \\
\mathcal{F}_3(z) &= F_1(z)F_1(z^2) \\
\end{align*}
\] 

(6.2)

Based on the design of section 3 and using (6.1) and (6.2) we can obtain a high quality, perfect reconstruction, FIR 4-channel filter bank. The frequency response of the above filters with \(H_0(z), H_1(z), F_0(z),\) and \(F_1(z)\) obtained in section 3 are illustrated in figure 6.1

![Figure 6.1 Frequency response of 4-channel filter bank](image)

Result of the simulation for different filter banks can be seen in the following sub-sections.
6.1. FIR filter with $H_1(z) = H_0(-z)$

As shown in figures 6.2 and 6.3 and table 6.1, most of the energy is still in the very lower part of frequency spectrum. So these images can be highly compressed by assigning different number of bits per pixel to each sub-band. The relation between energy level and number of pixels can be found in figure 6.4.

Figure 6.2 – Sub-bands of Image using filter banks designed in section 2 (all but first one are gained by 10 to make them more visible)
Figure 6.3 – Sub-bands of Image using filter banks designed in section 2
(all but first one are gained by 10 to make them more visible)

<table>
<thead>
<tr>
<th></th>
<th>Lenna</th>
<th>Baboon</th>
</tr>
</thead>
<tbody>
<tr>
<td>98.6316%</td>
<td>0.2289%</td>
<td>0.0532%</td>
</tr>
<tr>
<td>0.5516%</td>
<td>0.0987%</td>
<td>0.0223%</td>
</tr>
<tr>
<td>0.1179%</td>
<td>0.0236%</td>
<td>0.0087%</td>
</tr>
<tr>
<td>0.1112%</td>
<td>0.0414%</td>
<td>0.0117%</td>
</tr>
</tbody>
</table>

Table 6.1 – Energy percentage of different sub-bands divided by filter designed in section 2
6.2. FIR filter with no constraint

The results of simulation can be seen in figures 6.4 and 6.5 and table 6.2. In figure 6.6 you can see images coded with only 1 bit per pixel.

**Figure 6.4** – Sub-bands of Image using filter banks designed in section 3
(all but first one are gained by 10 to make them more visible)
**Figure 6.5** – Sub-bands of Image using filter banks designed in section 3 (all but first one are gained by 10 to make them more visible)

<table>
<thead>
<tr>
<th>Lenna</th>
<th>Baboon</th>
</tr>
</thead>
<tbody>
<tr>
<td>98.4035%</td>
<td>96.3171%</td>
</tr>
<tr>
<td>0.1701%</td>
<td>0.7574%</td>
</tr>
<tr>
<td>0.0187%</td>
<td>0.2053%</td>
</tr>
<tr>
<td>0.0438%</td>
<td>0.5564%</td>
</tr>
<tr>
<td>0.4377%</td>
<td>0.4789%</td>
</tr>
<tr>
<td>0.1224%</td>
<td>0.3047%</td>
</tr>
<tr>
<td>0.0064%</td>
<td>0.1153%</td>
</tr>
<tr>
<td>0.0313%</td>
<td>0.2497%</td>
</tr>
<tr>
<td>0.0334%</td>
<td>0.0509%</td>
</tr>
<tr>
<td>0.0100%</td>
<td>0.0315%</td>
</tr>
<tr>
<td>0.0046%</td>
<td>0.0244%</td>
</tr>
<tr>
<td>0.0089%</td>
<td>0.0468%</td>
</tr>
<tr>
<td>0.1126%</td>
<td>0.1860%</td>
</tr>
<tr>
<td>0.0462%</td>
<td>0.1230%</td>
</tr>
<tr>
<td>0.0084%</td>
<td>0.0632%</td>
</tr>
<tr>
<td>0.0252%</td>
<td>0.1169%</td>
</tr>
</tbody>
</table>

**Table 6.2** – Energy percentage of different sub-bands divided by filter designed in section 3
Figure 6.6 – Output images coded only using in average 1.0 bit per pixel